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ScienceDirect

Historia Mathematica 62 (2023) 3–18

HISTORIA MATHEMATICA

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How the estimate of $\sqrt{2}$ on YBC 7289 may have been calculated

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Available online 21 October 2022

Abstract

It remains unknown how the approximation of $\sqrt{2}$ scribed on Babylonian tablet YBC 7289 was calculated. In this article I show how it can be straightforwardly computed using a well-known regular number as the input for the Babylonian method of estimating square roots. My objective is to demonstrate that Babylonian mathematics was sufficiently evolved for the approximation to be easily derived and thus propose an approach that may have been used to calculate it.
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Résumé

On ne sait toujours pas comment l'approximation de $\sqrt{2}$ inscrit sur la tablette babylonienne YBC 7289 a été calculée. Dans cet article, je montre comment elle peut être calculée simplement en utilisant un nombre régulier bien connu comme entrée pour la méthode babylonienne d'estimation des racines carrées. Mon objectif est de démontrer que les mathématiques babyloniennes étaient suffisamment évoluées pour que l'approximation puisse être facilement dérivée et proposer ainsi une approche qui aurait pu être utilisée pour la calculer.
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MSC: 10.010; 20.030

Keywords: Square root of two; YBC 7289; Regular number; Reciprocal; Babylonian method

1. Introduction

If the history of mathematics were a story, $\sqrt{2}$ would be its villain. Arithmetic would be a vastly more straightforward exercise if every number could be expressed as a fraction. However, by 400 BCE, Greek

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[1] Unaffiliated independent researcher. This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

https://doi.org/10.1016/j.hm.2022.08.002
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mathematicians had proved that $\sqrt{2}$ could not be so expressed and, instead, can only be estimated to some finite precision.[2]

The focus of this article is to examine the well-known, and very accurate, Old Babylonian estimate of $\sqrt{2}$ scribed on tablet YBC 7289 (c. 1800 – 1600 BCE).

Historians of cuneiform mathematics have made great strides in deriving many Babylonian mathematical techniques from archaeological findings. Unfortunately, there is no extant artefact that proves how the Babylonians calculated their estimate of $\sqrt{2}$. Instead, we must rely on the conjecture of mathematicians who pay due heed to the scholars of Babylonian history and language.[3]

In this article I will explore how $\sqrt{2}$ might have been estimated using solely documented mathematical practices from the Old Babylonian period. I propose one approach which generates the estimate on YBC 7289 with very few calculations.

For comparison purposes I also present anachronistic and unlikely approaches that the Babylonians could have used to estimate $\sqrt{2}$. This shows the relative simplicity of my proposed approach and highlights the pitfalls of proposing calculation without familiarity with the culture.

2. Babylonian numbers

Before proceeding, even with describing YBC 7289, an understanding of the number system used in the Old Babylonian period is required. It is widely accepted[4] that Babylonian mathematicians:

- Used a positional sexagesimal (base 60) number system, commonly called *sexagesimal place value notation* (henceforth abbreviated SPVN).
- Did not use a radix character (e.g. a decimal point).
- Had no symbol for zero.

I will follow the convention of Chemla and Michel (2020, p. 537) and represent an SPVN number as a series of colon-separated decimal numbers between 1 and 59, each representing a single Babylonian sexagesimal digit. For example, 1:25 in SPVN is 85 in decimal ($1 \times 60 + 25$).

I will not use a radix character which means the reader will need to infer the units of the Babylonian numbers written in this article. For example, 1:25 in SPVN also equals decimal 5100 ($1 \times 60^2 + 25 \times 60$); decimal 1.41666... ($1 + 25/60$); and, more generally, decimal 85×60^n for any integer n.

Without a radix character, SVPN offers no notion of magnitude (Proust, 2012, p. 403). However, in context, the Babylonians did have a notion of one number being less than or greater than another, regardless of the number of digits that constitute those numbers. For example, the scribe of Old Babylonian text M10 records that 8:34:16:59 is less than a particular reference number, and 8:34:18 is greater than it (Sachs, 1952, p. 152). Since today we would express this reference number as 60/7 (= 8.571428 recurring), modern day readers might intuit the scribe's ordering of 8:34:16:59 < 8:34:18 by considering the two numbers as if a "decimal point" appeared after the initial digit 8. In the same way, a scribe working in the context of $\sqrt{2}$ would, and we shall henceforth, order a number such as 1:59:59:59 as less than 2. In this article I will use terms of order-relation, such as *bigger* and *smaller*, and *greater than* and *less than*, which should be

[2] No firm date is known for when the irrationality of $\sqrt{2}$ was proved, however the irrationality of the square roots of higher primes is referenced in Plato's *Theaetetus* (c. 369 BCE) as being known by Theodorus of Cyrene (465 BCE – 398 BCE) implying that the irrationality of $\sqrt{2}$ was already known by then.

[3] In refuting claims that the tablet Plimpton 322 was a trigonometric table Robson (2001, p. 167) posited that one cannot interpret tablets from mathematical intuition alone but that one must explore them in the context of Babylonian culture.

[4] See, for example, Neugebauer and Sachs (1945, p. 2), Bruins and Rutten (1961, p. 10), Friberg (2007, p. 5), Høyrup (2002a, p. 12), Proust (2007, p. 74).

interpreted in the same way as the order-relation terms were intended in M10 (which Sachs translated as *deficit* and *excess*). To avoid any ambiguity, when referring to the size of a number in terms of its number of digits I will use the terms *longer* and *shorter*. For example, 8:34:16:59 is longer than 8:34:18, and 1:59:59:59 is longer than 2.

Generally, I will not use the digit 0. The main exception is when, for example, denoting the number 10 which should be considered as a single symbol for *the number after 9* rather than *1 ten and 0 units*. The same applies to 20, 30, 40, and 50. Elsewhere, when zero does appear in my synthesis, its use is qualified.

3. YBC 7289

YBC 7289 is a circular Old Babylonian clay tablet of about 7 cm in diameter dated c. 1800 – 1600 BCE.[5] On the obverse is the drawing of a square, with two intersecting lines in its interior, each joining the opposite corners of the square. Along one of these diagonal lines is the number 1:24:51:10.

Of secondary interest, the number 30 appears on the exterior of the square, and elsewhere on the interior is the number 42:25:35.

The feature that makes YBC 7289 such an important artefact is that 1:24:51:10 is, in fact, an Old Babylonian estimate of $\sqrt{2}$. In decimal, with the decimal point suitably positioned, this estimate is 1.414213 which is accurate to 6 significant figures. This astonishing level of accuracy would have been unachievable by physically drawing and measuring the diagonal. Instead, it must have been calculated.

In the remainder of this article, I will show that documented mathematical practices from the Old Babylonian period can be used to construct this estimate of $\sqrt{2}$ in very few steps.

4. The Babylonian method of estimating square roots

The Babylonian method for approximating square roots is well documented. Its use continues to this day, and mathematicians refer to it as the *Babylonian method*. Fowler and Robson (1998, p. 371) cited an example of its use in tablets BM 96957 + VAT 6598. Written in modern algebra, the *Babylonian method* is this:

> If the lengths of the two short sides of a right-angle triangle are a and b, then the length of the hypotenuse can be approximated by $a + 1/2\ b^2/a$.

This approximation increases in accuracy as b decreases. Therefore, if b represents the error in an initial estimate, a, of the square root of a number, d, then today we write the Babylonian estimate as

$$1/2\ (a + d/a) \tag{1}$$

In fact, this *Babylonian method* of estimating $\sqrt{d}$ is equivalent to Sir Isaac Newton's root[6] finding method, applied to the square function, and therefore we know the error of the Babylonian estimate is quadratically smaller than the error of its initial estimate (see for example Epperson, 2013). Consequently, today the *Babylonian method* is used recursively (i.e., using the resultant estimate from one iteration as the initial estimate in the next) to rapidly calculate square roots to any desired level of accuracy.

[5] The shape and size of this tablet suggest it is a Type IV tablet (Middeke-Conlin, 2020, pp. 37-39), which are almost always exercises in advanced calculation (Proust, 2007, p. 90) in a pedagogical setting (Fowler and Robson, 1998, p. 369).

[6] A root of a mathematical function is the value for which the function equals some specified target. The square root of d is simply the root of the function $x^2 = d$.

Taking $d = 2$ in our formula above, the *Babylonian method* to estimate $\sqrt{2}$ is

$$a/2 + 1/a \quad (2)$$

where a is the initial estimate.

5. Babylonian arithmetic

It is well attested[7] that although Babylonian mathematicians used addition, subtraction, and multiplication much as we do today, division did not exist. Therefore, the Babylonian computation of, what we have termed, $1/a$ in equation (2) was problematic.

Rather than dividing by a number, Babylonian mathematicians multiplied by its reciprocal (footnote 7). In modern algebra, the reciprocal of a number a, denoted by $\overline{a}$, satisfies

$$a \times \overline{a} = 1 \quad (3)$$

so, we can consider $\overline{a}$ as equal to $1/a$ (since SVPN numbers have no radix character, the number 1 in equation (3) could be 1, 60, 60^2, 60^3 and so on). Therefore, $\overline{a}$ can be used in place of $1/a$ in the *Babylonian method*. However, not every number has a reciprocal that can be expressed as a sexagesimal number of finite length. Those that do are termed *regular* numbers (Neugebauer, 1935, p. 5).

Regular numbers lie at the heart of the debate about how $\sqrt{2}$ could have been found.

6. Elementary regular numbers

Since regular numbers could be used as divisors, they were especially important to the Babylonians. It has been hypothesised, from as early as Theon of Alexandria circa 400 AD, that the base 60 number system was used specifically because of its large proportion of regular numbers: 25 of the 60 single-digit sexagesimal numbers are regular (Proust, 2007, p. 199). Neugebauer and Sachs (1945, p. 11) observed that regular number are those that only have prime factors 2, 3, and 5. Therefore, 1, 2, 3, 4, 5, 6, 8, 9, and 10 are all regular numbers for example.

Numerous cuneiform tablets (such MS 3874) list the single-digit regular numbers and their reciprocals, known as the *elementary* regular numbers (Proust, 2012, p. 391).[8]

Tables of reciprocals played a central role in Babylonian arithmetic, and were memorised during scribal education (Proust, 2007, pp. 118-128). Such was their importance that, in the scribal curriculum of Nippur reciprocal tables were learned before any other numerical table, even multiplication tables (Proust, 2007, pp. 150-152).

7. Using the Babylonian method to construct a two-digit estimate of $\sqrt{2}$

Having introduced regular numbers and their reciprocals, we can compute $1/a$ in equation (2). By selecting a to be a regular number I can now demonstrate the *Babylonian method*. I do this in a small example replicating the first part of the proposal of how the number on YBC 7289 was derived by Neugebauer and Sachs (1945, p. 43). They started with the elementary regular number 1:30 (1.5 in decimal). Half of 1:30 is

[7] See, for example, Neugebauer (1935, p. 5), Neugebauer and Sachs (1945, p. 11), Bruins and Rutten (1961, p. 10), Friberg (2007, p. 8), Høyrup (2002a, p. 28), Proust (2007, p. 119).

[8] Some tables also include the double-digit numbers 1:04 and 1:21 as elementary regular numbers.

45, and the reciprocal of 1:30 is 40, therefore, one iteration the *Babylonian method* yields the estimate of $\sqrt{2}$ as 45 + 40 which is 1:25 (1.41666... in decimal).

Tablet TMS III contains a list of mathematical constants and Bruins and Rutten (1961, p. 26) noted that 1:25 appears on it as the length of the diagonal of the unit square (i.e., $\sqrt{2}$), proving this estimate of $\sqrt{2}$ was used by the Babylonians.

We can assess if an estimate of $\sqrt{2}$ is too big, or too small, by squaring it and comparing it to 2 (in the same way that the scribe of M10 did not need to know the exact reciprocal of 7 but could assess if his estimate was too big or too small by multiplying it by 7 and seeing if the result was greater than, or less than, 1). The square of 1:25 is greater than 2. Clearly then,1:30 was too high an initial estimate. Therefore, as a second example of the application of the *Babylonian method*, I use the regular number 1:20 (1.333... in decimal) as an initial estimate, whose square is less than 2. Half of 1:20 is 40 and the reciprocal of 1:20 is 45, their sum will also produce an estimate of $\sqrt{2}$ as 1:25.

We should note a specific pattern in these two examples. The first example began with an initial estimate greater than $\sqrt{2}$, and the second example with an initial estimate less than $\sqrt{2}$, and yet in both cases the resultant estimate, 1:25, was greater than $\sqrt{2}$. In fact, the geometric explanation of the *Babylonian method* (such as that in Fowler and Robson, 1998, p. 371) proves that the resultant estimate of the square root will always be too big. Geometric proof notwithstanding, this pattern would perhaps have been empirically obvious to anyone using the *Babylonian method* on a regular basis.

This over-estimation property of the *Babylonian method* has a useful implication:

- if the resultant estimate of a square root is less than the initial estimate, then the initial estimate was greater than the square root,
- if the resultant estimate of a square root is greater than the initial estimate, then the initial estimate was less than the square root.

This will prove useful in estimating $\sqrt{2}$.

Although the *Babylonian method* is used today by taking as many recursive iterations as are necessary to meet the desired level of accuracy, that is not the case for the Babylonian mathematicians. Recall that the method requires division by the initial estimate, a. Therefore, in the Babylonian mathematical paradigm, a must be a regular number. Once the *Babylonian method* outputs an irregular number, no further iterations of recursion can occur. This usually happens after one iteration as we saw when generating the irregular number 1:25 as an estimate of $\sqrt{2}$.

8. Longer regular numbers

In order to use the *Babylonian Method* to discover a better estimate of $\sqrt{2}$ than 1:25, a longer regular number (i.e. consisting of more digits) is required as an initial estimate. Since 1:25 has been generated by the *Babylonian Method* we know it is greater than $\sqrt{2}$, and therefore a longer regular number marginally less than 1:25 would be ideal for this purpose.

There are many texts demonstrating that the Babylonians understood that the product of regular numbers was also a regular number, whose reciprocal would be the product of the reciprocals of the factors.[9] The Babylonians used this fact to construct algorithms to derive, and identify, bigger regular numbers – and their reciprocals.

The instructions of one such algorithm, which Sachs (1947, p. 223) termed *The Technique*, appear on tablet VAT 6505. The algorithm works by removing the trailing digits of a multi-digit number, which

[9] See, for example, Neugebauer (1935, pp. 23-24), Friberg (2007, Sec. 1.4), Proust (2007, Sec. 6.2-6.3).

themselves constitute an elementary regular number. The remaining digits are multiplied by the reciprocal of this regular number and 1 is added. The result is examined to see which of its trailing digits are also an elementary regular number. This process is repeated, with the length of the resultant number reducing each iteration, until the initial multi-digit number has been factorised into elementary regular numbers.

There is evidence of the widespread use of *The Technique*; for example, tablet CBS 1215 contains 21 applications of it to find reciprocals of large numbers.

As a straightforward example of *The Technique*, consider the number 22:30. The trailing digit 30 is an elementary regular number, whose reciprocal is 2. We calculate $22 \times 2 + 1 = 45$, which is an elementary regular number, whose reciprocal is 1:20. Therefore, 22:30 is regular and equals 45×30. Furthermore, the reciprocal of 22:30 is $2 \times 1{:}20 = 2{:}40$.

I have selected this example because Proust (2012, p. 403) noted that the regular pair 22:30 and 2:40 were commonly known and therefore used in exercises deploying *The Technique* as if they were elementary numbers.

The prevalence of the number 22:30 on mathematical tablets is evidence that it was well known by the Babylonians.[10] Proust (2007, pp. 128-133; 2012, p. 403) contended that the multiplication table of 22:30 was learnt by heart at primary schools, such was the familiarity of this regular number to the Babylonians. That multiplication table would show that any multiple of 3 appended to 22:30 will produce a number that is itself a multiple of 22:30. Since 1:24 is a multiple of 3, a scribe wanting a good regular number for estimating $\sqrt{2}$ and consequently hunting for a regular number beginning 1:24, might therefore consider 1:24:22:30. A single iteration of *The Technique*, would confirm to the scribe that $1{:}24{:}22{:}30 = 3{:}45 \times 22{:}30$ and since 3:45 appears on the standard table of reciprocals as the reciprocal of 16, the number 1:24:22:30 would readily be recognised as regular.

In fact, 1:24:22:30 appears twice on CBS 1215 in applications of *The Technique*, expressed as the product of the familiar regular numbers 3:45 and 22:30.

A second algorithm is termed *the doubling and halving algorithm* by Friberg (2005, p. 18), leaning on the nomenclature of Sachs (1947, p. 224). The algorithm starts with an elementary regular number, recursively doubles it to generate a list of additional regular numbers; and recursively halves the reciprocal of this elementary regular number to generate the corresponding reciprocals.[11] For example, the regular number 5 can be recursively doubled to produce the elementary regular numbers 10, 20, 40, and 1:20; and the reciprocal of 5, which is 12, can be recursively halved to produce the corresponding reciprocals 6, 3, 1:30, and 45. Friberg (2005, p. 18) claimed the best-known example text of the doubling and halving algorithm is UM 29-13-21. Robson (2001, p. 193), Proust (2012, p. 407), and Neugebauer and Sachs (1945, p. 13)[12] all claimed that the righthand column of the obverse of UM 29-13-21 tabulates the continuation of the example I just cited, listing the two-digit regular numbers 2:40, 5:20, 10:40, 21:20, 42:40 and their corresponding reciprocals 22:30, 11:15, 5:37:30, 2:48:45, and 1:24:22:30.

Whilst not as familiar as the elementary regular numbers, the presence of 1:24:22:30 on both CBS 1215 and UM 29-13-21 suggests the Babylonians had some knowledge of it as a regular number. This may be due to its reciprocal being just a two-digit number 42:40 and therefore listed in a table of two-digit regular numbers, such as the table in Neugebauer (1935, §7 Anhang, p. 83).

Proust (2007, Sec. 6.3) believed that texts such as CBS 1215 and UM 29-13-21 are possibly reference texts for teaching purposes. She cited several other example texts in the style of these and hypothesised that

[10] See, for example, the numerous occurrences of 22:30 in the tablets analysed in Gonçalves (2015, pp. 28, 42, 68, 69) and Proust (2012, Table 12.6, p. 406) and its frequent use as a head number in lists of multiplication tables by Neugebauer and Sachs (1945, pp. 25-33) and Neugebauer (1935, p. 34).

[11] Since 2 is a regular number, the double and half of a regular number must also be regular.

[12] Neugebauer and Sachs (1945, p. 13) list UM 29-13-21 as CBS 29.13.21.

many copies of them (or like them) existed in Nippur. Proust concluded that regular numbers formed by a geometric series, such as 1:24:22:30, have a pedagogical utility and could have been widely available.

In a more applied setting, there are numerous texts related to brick deliveries and earth transportation (see, for example, Robson, 1999; Friberg, 2001; Middeke-Conlin, 2022). Mathematical exercises pertaining to bricks and mud use *nazbalum*, a 'carriage coefficient', which denotes the number of bricks a single worker can carry a specified distance in a specified time (Robson, 1999, p. 83). Different types of bricks were used, and each type of brick had its own carriage number. Robson (1999, p. 84) claimed that *monthly carriage numbers of bricks* were well-known to the Babylonians, being tabulated in coefficient lists. Neugebauer and Sachs (1945, p. 38) claimed that all such parameters "must be known by anyone dealing with various types of mathematical texts". Robson (1999, p. 84) listed 1:24:22:30 as the monthly carrying number of Type 9 bricks.[13]

I claim that all the above is strong evidence that 1:24:22:30 was a known regular number to the Babylonians.[14]

9. How the estimate of $\sqrt{2}$ on YBC 7289 might have been calculated

In this section I will show how the estimate of $\sqrt{2}$ on YBC 7289 could have been calculated. To illustrate its straightforwardness, I will provide every individual computation I executed to reach 1:24:51:10.

Proust (2000), Høyrup (2002a, p. 73; 2002b; 2007, p. 262; 2018), Woods (2017, p. 448) and Middeke-Conlin (2020, Chap. 6) believe that some kind of abacus, counting board, or scratch pad, was used by the Babylonians for individual arithmetic operations. Therefore, in this section, I will incorporate a '0' digit in the SPVN numbers, which should be interpreted by the reader as an empty placeholder on such a counting device. Also, I will use the arithmetic method of squaring proposed by Buckle (in preparation) which lends itself to a counting device (any other method of squaring may be used by the reader, if preferred).

Knowing that 1:25 marks the end of the recursive steps of the *Babylonian method* to estimate $\sqrt{2}$, scholars have wondered whether a regular number, close to 1:25 might be used to restart the *Babylonian method* to further improve the estimate (I detail this literature later in this article).

I suggest that 1:24:22:30 would have been a reasonable choice to the scribe of YBC 7289:[15] It is present in other mathematical computations of the same period; it is close to, but less than 1:25; it is regular; and its reciprocal, which will be needed in the computation of the Babylonian method, is only two digits long.

Moreover, critically in our context, Friberg (2001, p. 99) suggested the unusual calculation in the brick related mathematical exercise in IM 54538 is due to the *nazbalum* of the pertinent brick type, 1:24:22:30, being approximated by 1:30. Since 1:30 was the number Neugebauer and Sachs conjectured was used in the Babylonian method to form the approximation of $\sqrt{2}$ of 1:25, and 1:24:22:30 was known to the Babylonians, and the proximity of 1:24:22:30 to 1:30 was used in mathematical exercises, 1:24:22:30 is a likely candidate as a multi-digit regular number initial estimate in the *Babylonian method*.

The proposed computation of the *Babylonian method* of estimating $\sqrt{2}$ starting with 1:24:22:30 is as follows:

[13] Table 5.4 contains Robson's calculations of carriage numbers for 12 types of brick. She applied the same formula to each type, and 7 of her 12 computed carriage numbers are consistent with those that appear on texts containing coefficient lists. However, 1:24:22:30 has not been found on such texts and is therefore a hypothesis by Robson. It is nevertheless a hypothesis supported by Gonçalves (2015, p. 68) and Friberg (2001, p. 99).

[14] I do not intend 'known' to mean 1:24:22:30 was as well, and widely, known as the elementary regular numbers, but that scribes concerned with multi-digit regular numbers had knowledge of it.

[15] Fowler and Robson (1998, p. 370) suggested that 1:24:51:10 used on YBC 7289 was copied from a coefficient list. In that case the scribe mentioned here is the scribe of that coefficient list rather than of YBC 7289.

i) Halve 1:24:22:30 to give 42:11:15
ii) Add this to 42:40 (the reciprocal of 1:24:22:30) to give 1:24:51:15

Therefore 1:24:51:15 is the resultant estimate of $\sqrt{2}$.

Note that no intensive computation is required since steps i) and ii) above are both straightforward mental arithmetic exercises. Fowler and Robson (1998, p. 376) claimed that a more computationally burdensome approach of applying the *Babylonian method* was used by Babylonian mathematicians than my calculation above (although the outcome is the same). I am personally not persuaded by their argument but have nevertheless included their alternative approach in Appendix A. That appendix also contains my reasoning for not supporting Fowler and Robson's claim.

Regardless of how it is computed, using 1:24:22:30 as the initial estimate, the *Babylonian method* produces an estimate of $\sqrt{2}$ of 1:24:51:15, which we know will be an overestimate. Crucially, it is bigger than the initial estimate, indicating that the initial estimate was less than $\sqrt{2}$. I have therefore bracketed $\sqrt{2}$ between 1:24:22:30 and 1:24:51:15. Consequently, the first digit of $\sqrt{2}$ is 1 and the second digit is 24.

At this point we can check the accuracy of 1:24:51:15 as our estimate of $\sqrt{2}$ by squaring it.

I do this squaring using the arithmetic method suggested by Buckle (in preparation) which assumes a reference table of squares of single-digit numbers is available:

i) Start with the square of 1, which is 1.
ii) Append to 1 the square of 24, which is 9:36, giving us 1:9:36.
iii) Multiply 2 × 24 × 1, which is 48, and add it to 1:9:36 (ignoring the last digit). The result is 1:57:36. This is the square of 1:24.
iv) Append to the square of 1:24 the square of 51, which is 43:21, giving us 1:57:36:43:21.
v) Multiply 2 × 51 × 1:24, which is 2:22:48,[16] and add it to 1:57:36:43:21 (ignoring the last digit). The result is 1:59:59:31:21. This is the square of 1:24:51.
vi) Append to the square of 1:24:51 the square of 15, which is 3:45, giving us 1:59:59:31:21:3:45.
vii) Multiply 2 × 15 × 1:24:51, which is 42:25:30,[17] and add it to 1:59:59:31:21:3:45 (ignoring the last digit). The result is 2:0:0:13:46:33:45 which is the square of 1:24:51:15.

As an interim calculation, in step v) we found that the square of 1:24:51 equals 1:59:59:31:21. I have therefore bracketed $\sqrt{2}$ between 1:24:51 and 1:24:51:15. Consequently, the third digit of $\sqrt{2}$ is 51.

All that remains is to find the fourth digit, which must be less than 15. To do that I note that the square of 1:24:51:15 is 0:0:13 above 2 and the square of 1:24:51 is 0:0:29 below 2, therefore the fourth digit of $\sqrt{2}$ will be about two thirds of 15. The obvious candidate to test as the fourth digit is therefore 10.

The square of 1:24:51:10 can also be computed using Buckle's method as follows:

i) Start with the square of 1:24:51, which is 1:59:59:31:21.
ii) Append to the square of 1:24:51 the square of 10, which is 1:40, giving us 1:59:59:31:21:1:40.
iii) Multiply 2 × 10 × 1:24:51, which is 28:17:0,[18] and add it to 1:59:59:31:21:1:40 (ignoring the last digit). The result is 1:59:59:59:38:1:40 which is the square of 1:24:51:10.

[16] I used the well-known Babylonian identity that double the product of two numbers equals the sum of their squares less the square of their difference. The square of 51 and 1:24 have already been computed to be 43:21 and 1:57:36, and their sum is therefore 2:40:57. Then, 1:24 minus 51 is 33 whose square is 18:9. This square is subtracted from 2:40:57 to give 2:22:48.

[17] I was able to compute 2 × 15 × 1:24:51 using mental arithmetic alone as follows: 2 × 15 is 30 and reciprocation tells us that multiplying by 30 is the same as dividing by 2. Therefore 2 × 15 × 1:24:51 is half of 1:24:51 which is 42:25:30.

[18] I was able to compute 2 × 10 × 1:24:51 using mental arithmetic alone as follows: 2 × 10 is 20 and reciprocation tells us that multiplying by 20 is the same as dividing by 3. A third of 1:24 is 28 and a third of 51 is 17, therefore 2 × 10 × 1:24:51 is 28:17.

Note that to compute the square of 1:24:41:10 I only had to do a single two-digit addition.

To complete my estimation, I will replace 10 as the fourth digit with 11, 12, 13, and 14 and see if the squares of these new numbers are closer to 2.

Actually, it is unnecessary to do this computation. If I know the square of a number, but want the square of the next number, I need only add twice the original number to its square then add 1. Since twice 1:24:51:10 will be a four-digit number starting with a 2, and the square of 1:24:51:10 is 1:59:59:59:38:1:40, the sum of these must exceed 60 in the fourth digit. Not only does this bracket the estimate of $\sqrt{2}$, between 1:24:51:10 and 1:24:51:11 but without any computation it is clear 1:24:51:11 must be greater than 2 one position earlier than 1:24:51:10 is less than 2. Therefore, we find the closest four-digit sexagesimal number to $\sqrt{2}$ to be 1:24:51:10. This is the number on YBC 7289.

10. A rejection of some other possible approaches to derive the estimate of $\sqrt{2}$ in YBC 7289

There are numerous other methods that might have been used by the Babylonians to estimate $\sqrt{2}$ (and some have been suggested by various scholars), although, to my mind, they all have practical shortcomings.

Possibly 1:24:51:10 was achieved by way of a brute-force attack, trialling multiple four-digit numbers.[19] However, finding a four-digit estimate this way would be tremendously computationally intensive. Conventional bisection makes trial-and-error more efficient but would still require a lot of arithmetic operations:18 iterations of halving and squaring are required to find a four-digit sexagesimal estimate of $\sqrt{2}$ from the initial interval of 1 to 2, and 13 iterations are required from the tighter initial interval of 1:24 to 1:25. This is perhaps why the published suggestions for how 1:24:51:10 was derived all assume that subtler mathematical methods were deployed.

The Babylonian method of estimating square roots is significantly faster than bisection. Given Friberg's (2007, p. 10; 2014) evidence of the Babylonians' use of recursion to finesse estimates in problems similar to the finding of square roots, it is tempting to suppose that the estimate in YBC 7289 could have been reached by recursively applying the Babylonian method for estimating square roots, as is commonplace today.

Neugebauer and Sachs (1945, p. 43), the original publishers of the tablet, proposed that the estimate on YBC 7289 might have been found in this way, using a recursive averaging (which is the same as the Babylonian method), with an initial estimate of 1:30. Their first iteration yielded 1:25 and their second iteration, when truncated to four digits, produces the number on YBC 7289. However, the truncation is needed because 1:25 is irregular and therefore its reciprocal, which is used in this method, has infinite digits in SPVN. Neugebauer and Sachs gave no explanation of how they dealt with the division by an irregular number. They did concede (in a footnote on page 43) that if it were done by the post-Renaissance technique of *continued fractions*, seven iterations would be needed which would require significant computation.

Furthermore, I previously showed the square of 1:24:51:10 is less than 2. Since we know that the *Babylonian method* always overestimates, 1:24:51:10 could not have been generated by that method, regardless of the initial estimate used. It might be suggested that 1:24:51:10 is a truncation of the output of the Babylonian method with a very-many-digit initial estimate. I believe this to be unlikely. Of numbers consisting of more than four digits, the only regular number of less than 7 digits that begins 1:24 or 1:25 is 1:25:25:46:52:60. Even if this number were known to the Babylonians, using it and its reciprocal 42:8:23:42:13:20 results in an estimate of $\sqrt{2}$ of 1:24:51:17:8:26:20. After truncation, this does not yield 1:24:51:10.

Fowler and Robson (1998, pp. 374-375) suggested that, hypothetically, the reciprocal of 1:25 in the Babylonian method could have been approximated but pointed out that evidence of such irregular number approximation is sparse. This claim is supported by Sachs (1952, p. 151) who wrote that "division by

[19] On a personal note, the intrigue we modern mathematicians have with our Babylonian forebears is the sophistication of their mathematics. I prefer to believe that such a mathematically evolved society would baulk at using a brute-force to estimate $\sqrt{2}$.

irregular number ... is rarely encountered in Babylonian mathematics". Fowler and Robson (1998, p. 375) noted that YBC 10529 lists estimates of the irregular numbers from 56 to 1:20 and some are to four-digit accuracy. Plausibly such an estimate for the reciprocal of 1:25 existed, or could have been derived. It would only need to be accurate to three digits to recover the estimate of $\sqrt{2}$ on YBC 7289 using the *Babylonian method*. However, I would add to Fowler and Robson's hypothesis that the YBC 7289 approximation was not derived by way of an approximation of the reciprocal of 1:25 because, even with modern mathematical methods, it is so computationally intensive to make such a sufficiently accurate approximation of this number that it is faster to estimate $\sqrt{2}$ directly using trial-and-error.

As an alternative to estimating reciprocals, Fowler and Robson (1998, p. 375) suggested that a nearby regular number to 1:25 could have been used as an initial estimate in the Babylonian method to yield an improved estimate of $\sqrt{2}$. Their resistance to this proposition was threefold:

i) no two-digit regular number is near enough to 1:25 to yield a materially better approximation than 1:25.
ii) the nearest three-digit regular number is 1:25:20 which is greater than 1:25 and thus further from $\sqrt{2}$.
iii) the use of a four-digit regular number was unlikely since four-, and more-, digit tables of reciprocals did not appear until the Seleucid era, after YBC 7289 was scribed.

Regarding point ii), Fowler and Robson (1998, p. 375) found that using 1:25:20 as the initial estimate in the Babylonian method yielded the estimate 1:24:51:15. This is the same number I computed when using 1:24:22:30 as the initial estimate. However, given their initial estimate was greater than $\sqrt{2}$, for Fowler and Robson 1:24:51:15 represented nothing more than a closer estimate, but still an overestimate, of $\sqrt{2}$ and they dismissed it accordingly. Whereas because my initial estimate of 1:24:22:30 was less than $\sqrt{2}$, I had bracketed $\sqrt{2}$ between 1:24:22:30 and 1:24:51:15 and it was this that led to my fast derivation of the estimate on YBC 7289.

Regarding point iii), although entire tables of four-digit regular numbers did not appear until after YBC 7289 was scribed, specific four-digit numbers were known prior to that. I argue that 1:24:22:30 is an easily constructed regular number, and there is evidence of its use in the Old Babylonian epoch.

Høyrup (2002a, p. 263) pointed out that the *side-and-diagonal numbers* algorithm, documented by Italian mathematician Leonardo Fibonacci in the Middle Ages, is geometrically derived with concepts that the Babylonians knew. Therefore, Høyrup argued that this algorithm could have been used by a scribe to estimate $\sqrt{2}$. He found that this algorithm generated the estimate of $\sqrt{2}$ to be the ratio of 239 and 169. If this ratio were expressed in SVPN, Høyrup observed that it would equal 1:24:51 plus a remainder of 21/169. He argued that this fraction is close to 1/8 which is close to 1/6, and the Babylonian reciprocal of 6 only has one sexagesimal digit, 10. Therefore, the four-digit estimate of $\sqrt{2}$ is 1:24:51:10. Whilst this estimate matches that on YBC 7289, Høyrup conceded that there is no evidence that the Babylonians knew of the side-and-diagonal numbers algorithm. I would add that although there is some evidence for the construction of fractions involving 1/2, 1/4, ..., 1/32 (see for example YBC 4607, problem 3) this relates to applied problems such as brick volumes as opposed to the abstract exercise of estimating $\sqrt{2}$. I am unaware of any evidence of the Babylonians dealing with fractions such as 21/169 nor approximating them with single digit reciprocals.

Joseph (2011, p. 147) followed the logic of Neugebauer and Sachs and showed that just one more iteration of the *Babylonian method* yields the decimal estimate 1.41421, and the SVPN representation of this number is the number on YBC 7298. However, Joseph makes no comment as to why his arithmetic is done in decimal with a radix character, nor the fact that 1.41421 is a truncation of the outcome of his final iteration which is actually 577/408 which has an infinite digit decimal representation, nor how a scribe might have converted 577/408 into SPVN since 408 is not a regular number, and thus its SVPN representation would also have infinite digits.

The modern approach of using additional terms in the Taylor expansion can immediately be ruled out because there is no evidence of such expansions before the 17^{th} century.

A method not mentioned in the extant body of research, but one which I considered in this study, is the possibility that the YBC 7289 figure was reached by estimating $\sqrt{1/2}$ and multiplying that by 2. The method is not without merit since, using the approach I describe in this article, it is even easier to estimate $\sqrt{1/2}$ than $\sqrt{2}$ (start with the estimate a = 42:40. Halve it to give 21:20. A quarter of its reciprocal, 1:24:22:30, is 21:5:37:30, computed by halving, then halving again. Adding these gives us 42:25:37:30. We find that the square of 42:25 is less than 1/2 and the square of 42:25:37 is greater than 1/2. Trying 42:25:36 and then 42:25:35 shows that the latter is the best three-digit estimate of $\sqrt{1/2}$). We know that 2 $\times\sqrt{1/2}$ is $\sqrt{2}$ and therefore 2 × 42:25:35 = 1:24:51:10 which matches the estimate on YBC 7289. In one sense, estimating $\sqrt{1/2}$ in this way answers the objection of the use of four-digit regular numbers raised by Fowler and Robson.

I suspect that the approach of estimating $\sqrt{2}$ by doubling an estimate of $\sqrt{1/2}$ probably was not taken because, from a mathematical perspective, using 1:24:22:30 is a more obvious first step when encountering the obstacle of the irregularity of 1:25 during recursion. Furthermore, Fowler and Robson (1998, p. 370) claimed that the $\sqrt{2}$ estimate of 1:24:51:10 was not computed on YBC 7289 but taken from a list of mathematical constants and that tablet YBC 7243, exemplifying such a list, contains the entry "1 25 51 10, the diagonal of the square". The number 42:25:35, as an approximation of $\sqrt{1/2}$, does not appear on YBC 7243.

Moreover, this approach assumes that the Babylonians knew the theory of surds, specifically that $\sqrt{2}$ equals 2 × $\sqrt{1/2}$. I am not aware of any evidence of this. However, it does prompt a question regarding the remaining numbers on YBC 7289 which I address in the next section.

11. The other numbers on YBC 7289

Fowler and Robson (1998, p. 369) suggested that the other numbers on YBC 7289 were an exercise for a student to find the length of the square's diagonal if the side is of length 30. The other number on YBC 7289 is the correct answer to that exercise: 42:25:35. Fowler and Robson had a difference of opinion as to whether the figure in the exercise represents the integer 30 being a practical application of the mathematics of squares inscribed in 60-unit squares (Robson's view), or whether it represents one-half (30/60) in which case 42:25:35 represents the reciprocal of $\sqrt{2}$ (Fowler's view). In geometry the reciprocal of $\sqrt{2}$ can see as much use as $\sqrt{2}$ itself and would therefore be a useful number in its own right.

If Robson's view is correct, it would be an ideal practical example for a student because the instructor could then show (or set as an additional exercise) that the answer is also the reciprocal of $\sqrt{2}$. Given Friberg's (2007, p. 15) observation that "as in so many … cases in Old Babylonian mathematical texts, data that appears to be arbitrarily chosen on closer inspection turns out to be carefully constructed", it would, in my opinion, be an extraordinary coincidence, and an intellectual waste, if the number 30 had been selected as the example square side length, and then the relationship with the reciprocal of $\sqrt{2}$ not explored. Therefore, I personally see no conflict in the difference of opinion, as Robson's scenario would surely have led to Fowler's, with the execution of one additional step. Nevertheless, I find myself wondering if the demonstration that $(\sqrt{2})$ / 2 = 1 / $(\sqrt{2})$ on YBC 7289 is perhaps an early application of surds. Certainly, the problems on Tablet BM 15285, which Fowler and Robson (1999, p. 369) described as relating to squares obliquely inscribed in unit squares, get very close to the concept of a surd. Mindful of Robson's (2001, p. 167) point discouraging mathematicians from making proposals of Babylonian mathematics exogenously from the context of Babylonian culture, I make no claim of this and instead pose it as a question for further study.

12. Conclusion

YBC 7289 is a well-known Babylonian tablet of huge significance to the history of mathematics. However, it remains unclear how the estimate of $\sqrt{2}$ contained in it was derived.

I showed that the regular number 1:24:22:30 was known to the Babylonians. It is close to, but less than $\sqrt{2}$ and as a regular number it can be used as an initial estimate in the *Babylonian method* to estimate $\sqrt{2}$. Doing so not only produces a good estimate of $\sqrt{2}$ but proves the first, second and third digits of $\sqrt{2}$ are 1, 24, and 51 respectively, and that the fourth digit must be less than 15. Using the error in this estimate the fourth digit is quickly found to be 10.

My approach to estimate $\sqrt{2}$ uses few computations:

- one iteration of the *Babylonian method* with a number whose reciprocal is only two digits
- one calculation to square a four-digit SVPN number, and
- an adjustment to that square in the final digit.

The computational burden of squaring four-digit SVPN numbers should not be underestimated as a deterrent when trying to calculate an accurate estimate of a square root. Fowler and Robson (1998, p. 375) acknowledged that "these calculations involve some substantial sexagesimal arithmetic", to the extent that it may be too tedious to contemplate making anything beyond two-digit estimates (using the *Babylonian method*). Nevertheless, I acknowledge that mathematical simplicity is not proof of the method used to estimate $\sqrt{2}$.

Instead, I assert that I have provided historical evidence that both the *Babylonian method* and the number 1:24:22:30 were accessible to the Babylonian mathematician. The speculative element of this article is whether they were used in combination by the mathematician to estimate $\sqrt{2}$. Without proof we can never know what approach was used, and any study with this objective must have an element of conjecture. My claim is that the Babylonians' familiarity with the methods constituting my proposed approach and the simplicity of its application must at least make this approach a credible candidate. Perhaps more importantly, to the history of mathematics and mathematical education, I showed that the estimate on YBC 7289 is surprisingly easy to derive using the tools available to the Babylonian mathematician.

Acknowledgments

The author wishes to thank two anonymous reviewers for the constructive suggestions of improvement. The author also expresses his sincere appreciation to Robert Middeke-Conlin who read several drafts of this manuscript and provided a huge amount of input for its theoretical strengthening and readability.

Appendix A. Using the more long-winded computation of the *Babylonian method*

Fowler and Robson (1998, p. 376) claim that the Babylonian estimate of $\sqrt{2}$ is actually

$$a + 1/2\,(2 - a^2)/a \tag{4}$$

This is algebraically identical to the equation I used,

$$a/2 + 1/a \tag{5}$$

but computationally much more tedious, especially as the number of digits of a increases.

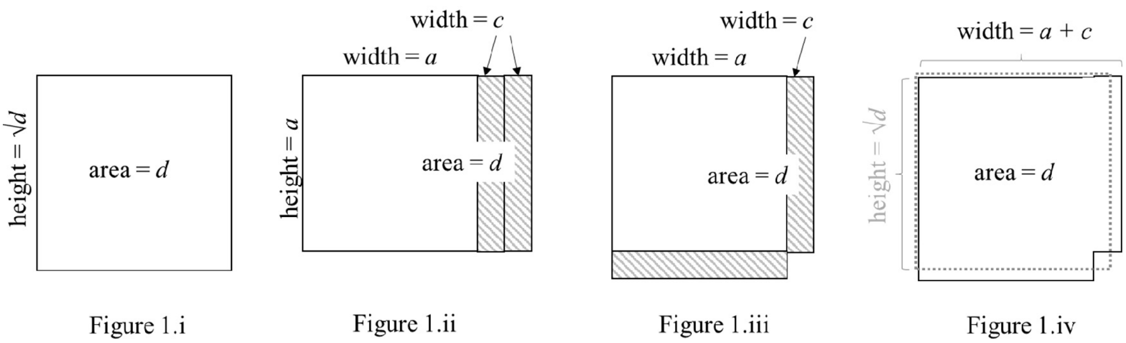

Figure 1. A geometric derivation of the Babylonian method of square rooting.

Fowler and Robson (1998, pp. 371-372) provide a geometric explanation of this square root estimation method which I will use here to explore how both equations can be derived. The geometry applies to the square root of any number, not just 2, and I will consider this more general case.

A.1. A geometric explanation of the Babylonian method to estimate the square root of some number, d

i) Begin with a square whose area is d (therefore its sides are of length $\sqrt{d}$) (Figure 1.i).
ii) If we have an estimate of $\sqrt{d}$, which we label a, then we can construct a rectangle that has height a and area d (the same area as the square).
iii) This rectangle can be considered as composed of a square of height a, and two identical rectangles of height a and width c (Figure 1.ii).
iv) Moving one of these identical rectangles to the bottom of the square makes a bigger square with the corner missing, but still with area d (Figure 1.iii). It has height and width equal to $a + c$.
v) If this missing corner has a small area (i.e. c is a lot smaller than a) then $a + c$ will approximate $\sqrt{d}$ (Figure 1.iv). Therefore, to estimate $\sqrt{d}$, we find the value of $a + c$.
vi) The rectangle in Figure 1.ii has area d, height a and width $a + 2c$. Therefore $a + 2c = d/a$. Adding another a and halving the result gives us $a + c = 1/2\ (a + d/a)$. This is the *Babylonian method* for estimating a square root.

Fowler and Robson differ from me in the final step of this geometric explanation. We both agree that we need to find $a + c$ from the equation $d = a^2 + 2ac$ where d and a are known and c is unknown. I did it by first finding $a + 2c$, which is just d/a, then adding another a and dividing by 2. Whereas Fowler and Robson first found c, which is $(d - a^2)\ /\ (2a)$ then added that to a. Their answer is the same as mine, but Fowler and Robson's method requires the burdensome computation of a^2 and its subtraction from d. The question is which of these methods did the Babylonian's use.

Fowler and Robson (1998, p. 371) cite BM 96957 + VAT 6598 (two fragments believed to be from the same tablet) as evidence of the Babylonian use of the *Babylonian method* of finding square roots. However, the problem on this tablet is a different, more complex, one. It is posed in the form:

A rectangle of height a, and width b. What is its diagonal?

The method followed in the tablet was this:

i) Square b.
ii) Divide b^2 by $2a$.
iii) Add that to a.

This can be summarised by saying the diagonal is approximately $a + 1/2\ b^2/a$.

Notice that there is no variable b in the geometry of Figure 1. BM 96957 + VAT 6598 is a different, albeit related, problem.

The crucial point is that, in the question posed on BM 96957 + VAT 6598, we do not know what number we are computing the square root of (i.e., we do not know d). We only know the height and breadth of the rectangle. To follow the geometric approach described above, the scribe of BM 96957 + VAT 6598 needed to, implicitly, identify d, and therefore relied on the Pythagorean Theorem that says $(\sqrt{d})^2 = a^2 + b^2$, giving us that $d = a^2 + b^2$. Then, the estimate of $\sqrt{d}$, which is $a + c$, can be found by noting that since the square in Figure 1.i has area equal to $a^2 + b^2$ then the two shaded rectangles in Figure 1.ii must have area b^2 in which case $c = 1/2\ b^2/a$. This is then added to a.

There is an important distinction between Fowler and Robson's geometric explanation of the *Babylonian method* and the method they cite from BM 96957 + VAT 6598: In the former we specify what square root we wish to estimate; in the latter we specify the height and breadth of a rectangle, and that implies what square root we will then be estimating.

Using the geometric concepts motivating the algorithm on this tablet, either $a + 1/2\ (d - a^2)\ /\ a$ or $a/2 + d/a$ could have been used to estimate $\sqrt{d}$. Only a tablet applying one of these formulae will inform us which was used. My point is that BM 96957 + VAT 6598 is not that tablet. Until such evidence is unearthed, we must rely on conjecture.

Fowler and Robson have implicitly conjectured the Pythagorean Theorem would be used in reverse to identify the value of b, from d and a, then the algorithm in BM 96957 + VAT 6598 could be applied. In my opinion, that assumption is questionable because it implies the Babylonians overlooked the simpler, but geometrically identical, method that I outlined, and which does not need the additional step of involving the Pythagorean Theorem. Additionally, it would result in an easier computation.

Furthermore, the factorisation required to get from $a + 1/2\ (d - a^2)\ /\ a$ to $1/2\ (a/2 + d/a)$ is identical to the factorisation used in *The Technique* and therefore the less tedious computational approach might have been known to the Babylonians even if the Fowler and Robson method was the conceptual starting point.

For these reasons, I have deferred the alternative computation of the Babylonian method for estimating $\sqrt{2}$ (i.e. $d = 2$) using Fowler and Robson's formula to this appendix. However, it is included for those readers who believe Fowler and Robson's formula was the more likely to have been used. Fortunately, when a = 1:24:22:30, the term $(2 - a^2)\ /\ a$ is only a two-digit number so the computations never get especially nasty.

A.2. Computing the Babylonian method applied using 1:24:22:30 using Fowler and Robson's equation

A summary of the computation of the *Babylonian method* using 1:24:22:30 is as follows:

i) Square 1:24:22:30 to give 1:58:39:8:26:15
ii) That is subtracted from 2:0:0:0:0:0 to give 1:20:51:33:45
iii) That is multiplied by 42:40 (the reciprocal of 1:24:22:30) to give 57:30
iv) That is halved to give 28:45
v) That is added to 1:24:22:30 to give 1:24:51:15

The third step is the only one that is computationally fiddly. Overall, it is fairly easy to compute.

However, there is a way of making this computation much easier. I point out that a is a regular number and therefore, using identical logic to that which motivated *The Technique*, $1/2\ (2 - a^2)\ /\ a$ can be factorised. This is especially relevant here since 1:24:22:30 appears on the tablets I cited earlier specifically because of its ease of construction from its factors (recall it is the reciprocal of 5×2^9). Were the scribe to execute a factorisation of $1/2\ (2 - a^2)\ /\ a$, its computation simplifies hugely to 1:15 × 23.

Using factorisation, my full calculation of the Babylonian estimate using 1:24:22:30 was as follows:

42:40 is factorised as 5×2^9, therefore its reciprocal, 1:24:22:30, must be factorised as $5^4 \times 3^5 \times 2$. Noting that 2×60^6 is factorised as $5^6 \times 3^6 \times 2^{13}$, we immediately see that $2 \times 60^6 - 1{:}24{:}22{:}30^2$ has common factor $5^6 \times 3^6 \times 2^2$, and therefore $1/2\ (2 - a^2)\ /\ a$ has common factor $5^2 \times 3$, which is 1:15.

After removing common factors, $2 \times 60^6 - 1{:}24{:}22{:}30^2$ equals $2^{11} - 5^2 \times 3^4$. The numbers 2^{11} and $5^2 \times 3^4$ are trivial to compute. 2^{11} is computed by doubling 2 ten times which is a simple mental arithmetic exercise, thus $2^{11} = 34{:}8$. We already know from the factorisation of 1:24:22:30 that $5^2 \times 3^2 = 3{:}45$ so it can be multiplied by 9 to find $5^2 \times 3^4$, which gives 33:45. Subtracting this from 34:8 gives 23 (this is $2 - a^2$ with all common factors removed). Multiplying that by 1:15 (the common factors of 2 and a^2) gives 28:45. To finish, this is added to 1:24:22:30 which gives the estimate of $\sqrt{2}$ of 1:24:51:15.

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